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LETTER TO THE EDITOR

Enhanced group velocity in metamaterials

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Abstract
The Bruggeman formalism is implemented to estimate the refractive index of an isotropic, dielectric, homogenized composite medium (HCM). Invoking the well-known Hashin–Shtrikman bounds, we demonstrate that the group velocity in certain HCMs can exceed the group velocities in their component materials. Such HCMs should therefore be considered as metamaterials.

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1. Introduction

By definition, metamaterials exhibit behaviour which (i) either their component materials do not exhibit (ii) or is enhanced relative to exhibition in the component materials [1]. Many types of metamaterials may be conceptualized through the process of homogenization [2–4], paving the way for their realization. For example, a homogenized composite medium (HCM) may be envisaged which supports the propagation of a Voigt wave (which is a planar wave whose amplitude varies linearly with propagation distance), although such waves cannot propagate through its component materials [5, 6].

In this communication, we explore the enhancement of group velocity which may be achieved through homogenization. Sølna and Milton recently considered this issue, by estimating the relative permittivity of a HCM as the volume-weighted sum of the relative permittivities of the component materials [7]. But that estimation is applicable only for planar composite materials such as superlattices of thin films, and not to the more commonly encountered particulate composite materials [2–4]. In the following analysis, we implement the well-established Bruggeman formalism [4] to calculate the effective refractive index of an isotropic dielectric HCM. Thereby, we demonstrate that metamaterials which support group velocities exceeding those in their component materials may be realized as particulate composite materials.

2. Analysis

Consider a composite material containing materials labelled a and b, with refractive indices n_a and n_b, respectively. The component materials are envisioned as random distributions of spherical particles. Provided that the diameters of these particles are small compared with electromagnetic wavelengths, homogenization techniques may be applied to estimate the effective refractive index of the HCM.

In particular, the well-established Bruggeman homogenization formalism [2, 4]—which may be rigorously derived from the strong-permittivity-fluctuation theory [8, 9]—leads to the equation

$$f_a \frac{n_a^2 - n_{\rm Br}^2}{n_a^2 + 2n_{\rm Br}^2} + f_b \frac{n_b^2 - n_{\rm Br}^2}{n_b^2 + 2n_{\rm Br}^2} = 0 \tag{1}$$

whose solution yields $n_{\rm Br}$ as the estimated refractive index of the HCM. Here, f_a and $f_b = 1 - f_a$ are the volume fractions of the component materials. In the following, both component materials are assumed to have negligible dissipation in the frequency range of interest.

The group velocity of a wavepacket propagating through the HCM is given as [10]

$$v_{\rm Br} = \left. \frac{c}{n_{\rm Br}(\omega) + \omega \frac{{\rm d}n_{\rm Br}}{{\rm d}\omega}} \right|_{\omega(k_{\rm avg})} \tag{2}$$

where $v_{\rm Br}$ is evaluated at the angular frequency $\omega = \omega(k_{\rm avg})$, with $k_{\rm avg}$ being the average wavenumber of the wavepacket, and c is the speed of light in free space. Similarly, the respective group velocities in component materials a and b are given by

$$v_\ell = \left. \frac{c}{n_\ell(\omega) + \omega \frac{{\rm d}n_\ell}{{\rm d}\omega}} \right|_{\omega(k_{\rm avg})} \qquad (\ell = a, b)\,. \tag{3}$$

We proceed to establish upper and lower bounds on $v_{\rm Br}$, in terms of n_a and n_b. In particular, we demonstrate that the inequalities

$$v_{\rm Br} > v_\ell \qquad (\ell = a, b) \tag{4}$$

can be satisfied for certain values of $n_a \geqslant 1$, $n_b \geqslant 1$, $\frac{{\rm d}n_a}{{\rm d}\omega} > 0$ and $\frac{{\rm d}n_b}{{\rm d}\omega} > 0$.

Differentiation of both sides of equation (1) with respect to ω yields

$$\frac{{\rm d}n_{\rm Br}}{{\rm d}\omega} = \delta_a \frac{{\rm d}n_a}{{\rm d}\omega} + \delta_b \frac{{\rm d}n_b}{{\rm d}\omega} \tag{5}$$

where

$$\left. \begin{aligned} \delta_a &= \frac{f_a n_a n_{\rm Br} \left(n_b^2 + 2n_{\rm Br}^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_{\rm Br}^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_{\rm Br}^2\right)^2} \\ \delta_b &= \frac{f_b n_b n_{\rm Br} \left(n_a^2 + 2n_{\rm Br}^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_{\rm Br}^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_{\rm Br}^2\right)^2} \end{aligned} \right\}. \tag{6}$$

Upper and lower bounds on δ_a and δ_b may be established by exploiting the Hashin–Shtrikman bounds n_L and n_U on $n_{\rm Br}$ [11]; i.e.,

$$n_L < n_{\rm Br} < n_U \tag{7}$$

where

$$\left. \begin{aligned} n_L^2 &= n_b^2 + \frac{3 f_a n_b^2 \left(n_a^2 - n_b^2\right)}{n_a^2 + 2n_b^2 - f_a \left(n_a^2 - n_b^2\right)} \\ n_U^2 &= n_a^2 + \frac{3 f_b n_a^2 \left(n_b^2 - n_a^2\right)}{n_b^2 + 2n_a^2 - f_b \left(n_b^2 - n_a^2\right)} \end{aligned} \right\}. \tag{8}$$

Combining equations (6)–(8), we get

$$\rho_\ell < \delta_\ell < \kappa_\ell \qquad (\ell = a, b) \tag{9}$$

where

$$\left.\begin{aligned} \kappa_a &= \frac{f_a n_a n_U \left(n_b^2 + 2n_U^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_L^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_L^2\right)^2} \\ \kappa_b &= \frac{f_b n_b n_U \left(n_a^2 + 2n_U^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_L^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_L^2\right)^2} \end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned} \rho_a &= \frac{f_a n_a n_L \left(n_b^2 + 2n_L^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_U^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_U^2\right)^2} \\ \rho_b &= \frac{f_b n_b n_L \left(n_a^2 + 2n_L^2\right)^2}{f_a n_a^2 \left(n_b^2 + 2n_U^2\right)^2 + f_b n_b^2 \left(n_a^2 + 2n_U^2\right)^2} \end{aligned}\right\}. \tag{11}$$

Thus, we have

$$\rho_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \rho_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega} < \frac{\mathrm{d}n_{\mathrm{Br}}}{\mathrm{d}\omega} < \kappa_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \kappa_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega} \tag{12}$$

and the group velocity in the HCM is accordingly bounded as

$$v_L < v_{\mathrm{Br}} < v_U \tag{13}$$

with

$$\left.\begin{aligned} v_L &= \frac{c}{n_U + \omega\left(\kappa_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \kappa_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega}\right)} \\ v_U &= \frac{c}{n_L + \omega\left(\rho_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \rho_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega}\right)} \end{aligned}\right\}. \tag{14}$$

If the inequalities

$$\left.\begin{aligned} n_U + \omega\left(\kappa_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \kappa_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega}\right) &< n_a + \omega \frac{\mathrm{d}n_a}{\mathrm{d}\omega} \\ n_U + \omega\left(\kappa_a \frac{\mathrm{d}n_a}{\mathrm{d}\omega} + \kappa_b \frac{\mathrm{d}n_b}{\mathrm{d}\omega}\right) &< n_b + \omega \frac{\mathrm{d}n_b}{\mathrm{d}\omega} \end{aligned}\right\} \tag{15}$$

hold for certain component materials, then the inequalities (4) are automatically satisfied.

The inequalities (15) reduce to the particularly simple inequality

$$n_U + \omega \frac{\mathrm{d}n_a}{\mathrm{d}\omega} \left(\kappa_a + \kappa_b - 1\right) + \kappa_b \left(n_a - n_b\right) < n_a \tag{16}$$

if $v_a = v_b$. The conditions

$$\left.\begin{aligned} n_U + \kappa_b \left(n_a - n_b\right) &< n_a \\ \kappa_a + \kappa_b - 1 &> 0 \end{aligned}\right\} \tag{17}$$

are satisfied, for example, by $n_a = 3$, $n_b = 1.2$ and $f_a = 0.9$. Thus, the inequality (16) holds, provided that the dispersive term $\frac{\mathrm{d}n_a}{\mathrm{d}\omega}$ is sufficiently small.

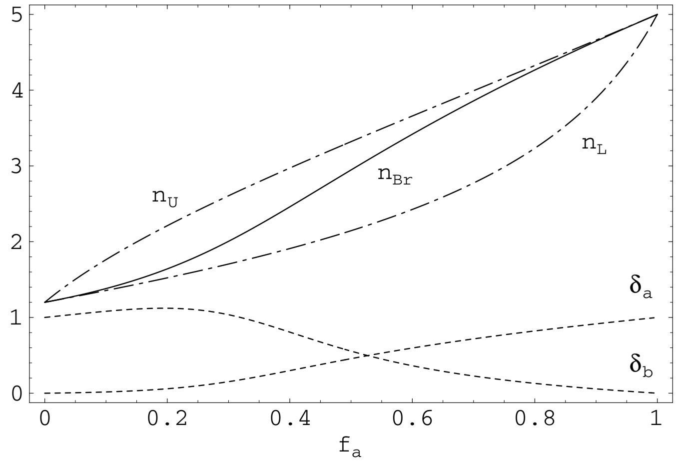

Figure 1. The estimated refractive index n_{Br} (solid line), the upper and lower Hashin–Shtrikman bounds on n_{Br} (broken dashed lines, labelled as n_U and n_L), and the coefficients δ_a and δ_b (dashed lines), all plotted as functions of the volume fraction f_a, when $n_a = 5$ and $n_b = 1.2$.

3. Numerical results

Let us illustrate the phenomenon represented by the inequalities (4) by means of a specific numerical example. Consider a particulate composite material at a particular value ω_0 of ω. At the chosen angular frequency, let $n_a = 5$, $n_b = 1.2$, $\frac{\mathrm{d}n_a}{\mathrm{d}\omega}\big|_{\omega=\omega_0} = 0.5/\omega_0$ and $\frac{\mathrm{d}n_b}{\mathrm{d}\omega}\big|_{\omega=\omega_0} = 5.5/\omega_0$. Significantly, material a has a high refractive index but low dispersion in the neighbourhood of ω_0, whereas high dispersion in material b is combined with a low refractive index.

The Bruggeman estimate of the refractive index of the HCM, namely n_{Br}, is plotted as a function of the volume fraction f_a in figure 1. Also shown are the upper and lower Hashin–Shtrikman bounds, n_U and n_L, on n_{Br}, as well as the parameters δ_a and δ_b. The Bruggeman estimate adheres closely to the lower bound n_L at low values of f_a, whereas at high values of f_a the difference between n_{Br} and its upper bound n_U becomes marginal. The observed agreement between n_{Br} and n_L at low f_a reflects the fact that the lower Hashin–Shtrikman bound is equivalent to the Maxwell Garnett estimate of the refractive index of the HCM arising from spherical particles of material a embedded in the host material b [4]. The Maxwell Garnett estimate is only valid then at low values of f_a. As the volume fraction becomes increasingly small, the Bruggeman estimate (n_{Br}) and the Maxwell Garnett estimate (low-f_a value of n_L) converge on n_b. In a similar manner, the agreement between n_{Br} and n_U at high values of f_a is indicative of the fact that the upper Hashin–Shtrikman bound is equivalent to the Maxwell Garnett estimate of the refractive index of the HCM arising from spherical particles made of material b embedded in host material a; the Maxwell Garnett estimate then holds only at high values of f_a. In the limit $f_a \to 0$, the coefficients $\delta_a \to 0$ and $\delta_b \to 1$; while $\delta_a \to 1$ and $\delta_b \to 0$ as $f_a \to 1$.

The corresponding group velocities v_a, v_b and v_{Br} are plotted as functions of f_a in figure 2. The upper and lower bounds on v_{Br} as given by v_U and v_L, respectively, are also displayed. Clearly, we have $v_{\mathrm{Br}} > v_a$ and $v_{\mathrm{Br}} > v_b$ for $f_a > 0.67$.

The inequalities (4) hold only over a relatively small range of parameter values. For example, the phase space in which the inequalities (4) are satisfied is illustrated in figure 3 for

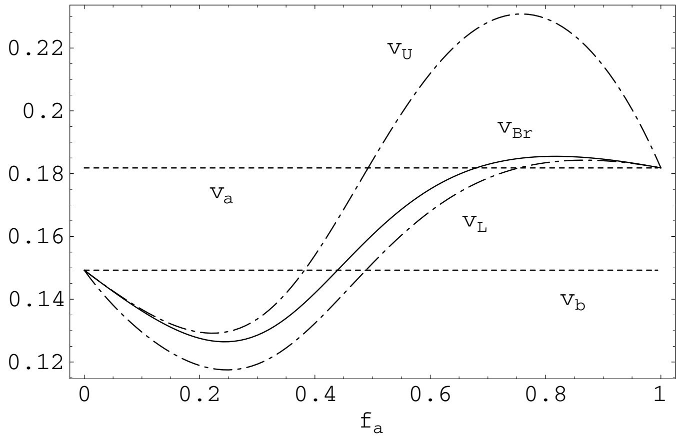

Figure 2. The estimated group velocity v_{Br} (solid line) and its upper and lower bounds (broken dashed lines, labelled as v_U and v_L), along with the group velocities v_a and v_b dashed lines) in the component materials, plotted as functions of the volume fraction f_a, when $n_a = 5, n_b = 1.2, \frac{\mathrm{d}n_a}{\mathrm{d}\omega}|_{\omega=\omega_0} = 0.5/\omega_0$ and $\frac{\mathrm{d}n_b}{\mathrm{d}\omega}|_{\omega=\omega_0} = 5.5/\omega_0$. All group velocities are normalized with respect to c.

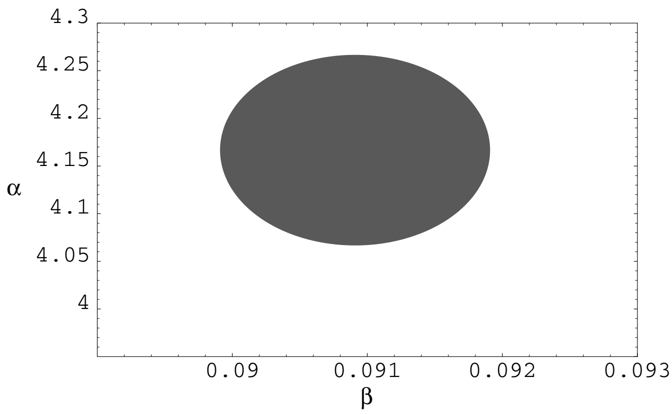

Figure 3. The shaded region indicates the portion of the α–β phase space wherein $v_{\mathrm{Br}} > v_a$ and $v_{\mathrm{Br}} > v_b$; here, $\alpha = \frac{n_a}{n_b}$ and $\beta = (\frac{\mathrm{d}n_a}{\mathrm{d}\omega} / \frac{\mathrm{d}n_b}{\mathrm{d}\omega})|_{\omega=\omega_0}$. This region was demarcated for $n_a = 5$, $\frac{\mathrm{d}n_a}{\mathrm{d}\omega}|_{\omega=\omega_0} = 0.5/\omega_0$ and $f_a = 0.8$.

$n_a = 5$, $\frac{\mathrm{d}n_a}{\mathrm{d}\omega}\big|_{\omega=\omega_0} = 0.5/\omega_0$ and $f_a = 0.8$. With these relationships fixed for the component material a, we find that $v_{\mathrm{Br}} > v_a$ and $v_{\mathrm{Br}} > v_b$ for

(i) $1.17 < n_b < 1.23$ with $\frac{\mathrm{d}n_b}{\mathrm{d}\omega}\big|_{\omega=\omega_0} = 5.51/\omega_0$ and
(ii) $5.45/\omega_0 < \frac{\mathrm{d}n_b}{\mathrm{d}\omega}\big|_{\omega=\omega_0} < 5.57/\omega_0$ with $n_b = 1.2$.

4. Concluding remarks

We conclude that the group velocity in an isotropic, dielectric, particulate composite material—as estimated via the Bruggeman homogenization formalism—can exceed the group velocities in its component materials. This metamaterial characteristic may be achieved through homogenizing (i) a component material *a* with high refractive index and low dispersion with (ii) a component material *b* with low refractive index and high dispersion. Neither anomalous dispersion nor an explicit frequency-dependent model of the refractive index (unlike [7]) is required to demonstrate this characteristic.

Improved estimates of HCM group velocity may be achieved through the implementation of homogenization approaches which take into better account the distributional statistics of the component materials (e.g., the strong-permittivity-fluctuation theory approach [8, 9]). In particular, the effects of coherent scattering losses—which are neglected in the present study—may well result in a moderation of the group velocity. Such studies are currently being undertaken, especially in light of the recent emergence of metamaterials wherein the phase velocity and the time-averaged Poynting vector are oppositely directed [12].

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